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Singular integral operators along surfaces of revolution

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Abstract

Let h, γ, and ϕ be radial functions on $\mathbb{R}^n$ and let $\Omega \in H^1(S^{n-1})$. Under certain natural conditions on γ and ϕ, we obtain L^p boundedness for the singular integral operators

$$T_{\alpha,\beta} f(x, x_{n+1}) = \text{p.v.} \int_{\mathbb{R}^n} h\bigl(|y|\bigr)\Omega(y')\, e^{i|y|^{-\beta}} |y|^{-n-\alpha} f\bigl(x - y, x_{n+1} - \gamma\bigl(|y|\bigr)\bigr)\, dy$$

and

$$Tf(x, x_{n+1}) = \text{p.v.} \int_{\mathbb{R}^n} h\bigl(|y|\bigr)\Omega(y')|y|^{-n} f\bigl(x - \phi\bigl(|y|\bigr)y', x_{n+1} - \gamma\bigl(|y|\bigr)\bigr)\, dy.$$

We also proved the L^p boundedness for the maximal operator associated with Tf.

1. Introduction

Singular integral operators have been studied by many mathematicians for several decades. The investigation of this interesting topic initially began with Calderón and Zygmund, and subsequently by Fefferman and several other well-known authors (for references, see [1,3,4,13,18–20] etc.). Recently, Fan and Pan

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(see [9]) have proved that, under certain growth conditions on ϕ, the singular integral operator

$$T_{\phi,h}f(x) = \text{p.v.}\int_{\mathbb{R}^n} f\big(x - \phi\big(|y|\big)y'\big)h(y)|y|^{-n}\Omega(y')\,dy \quad \big(\Omega \in H^1(S^{n-1})\big)$$

is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, and $n \geqslant 2$. Their work motivated us to seek for a positive answer about the L^p boundedness of the maximal operator associated with the above singular integral operator. Moreover, we would like to find certain natural conditions on ϕ which give the L^p bounds of those operators. As a consequence, we have proved the L^p boundedness of the singular integral operator $T_{\phi,h}f(x)$ and its associated maximal operator. We have also extended the results to operators along surfaces of revolution.

Another interesting class of singular integral operators is $T_{\alpha,\beta}f(x, x_{n+1})$ (as defined in the abstract). Integral operators with strong singularities at the origin were studied by Hirschman in one dimension (see [14]), Wainger in k dimensions (see [21]), Stein [17], Fefferman [11], and Fefferman and Stein [12]. Recently, Chandarana (see [2]) proved that for $\gamma(t) = |t|^k$ or $|t|^k \operatorname{sgn} t$, $k \geqslant 2$, and $\beta > 3\alpha > 0$, the singular integral operator

$$T_{\alpha,\beta}f(x_1, x_2) = \text{p.v.}\int_{-1}^{1} f\big(x_1 - t, x_2 - \gamma(t)\big)\frac{e^{-2\pi i|t|^{-\beta}}}{t|t|^{\alpha}}\,dt$$

is bounded on $L^p(\mathbb{R}^2)$ for

$$1 + \frac{3\alpha(\beta+1)}{\beta(\beta+1) + (\beta - 3\alpha)} < p < \frac{\beta(\beta+1) + (\beta - 3\alpha)}{3\alpha(\beta+1)} + 1.$$

We were motivated by Chandarana's work on this topic. This led us to investigate the minimal natural conditions on γ; and as a result, we have (in some sense) extended the class of γ. Moreover, due to the work done on $\Omega \in H^1(S^{n-1})$ by Fan, Pan, and several other authors (see [5,6,8], etc.). We were able to generalize the results on higher dimensions with $\Omega \in H^1(S^{n-1})$. We summarize our results below.

Theorem 1. *Let $\Omega \in H^1(S^{n-1})$ be homogeneous of degree zero, and have mean value zero over the sphere S^{n-1}, $n \geqslant 2$. Let h, γ, and ϕ be real-valued, measurable, and radial functions defined on $\mathbb{R}^n$.*

Assume that $h(t) \in L^\infty(\mathbb{R})$. Suppose that $\phi(t)$ is smooth on $(0, \infty)$, and for $t \in (0, \infty)$ and some $d > 0$,

(a) $|\phi(t)| \leqslant C_1|t|^d$,
(b) $C_2 t^{d-1} \leqslant |\phi'(t)| \leqslant C_3|t|^{d-1}$,

(c) $|\phi''(t)| \leqslant C_4|t|^{d-2}$.

If the one-dimensional maximal function

$$M^{\gamma} g(x_{n+1}) = \sup_{r>0} \left\{ \frac{1}{r} \int\limits_{|t|\leqslant r} \left| g\bigl(x_{n+1} - \gamma(t)\bigr) \right| dt \right\}$$

is bounded on $L^p(\mathbb{R})$ for all $p > 1$, then the singular integral operator

$$Tf(x, x_{n+1}) = \text{p.v.} \int \frac{h(|y|)\Omega(y')}{|y|^n} f\bigl(x - \phi\bigl(|y|\bigr)y', x_{n+1} - \gamma\bigl(|y|\bigr)\bigr)\, dy$$

and its associated maximal operator

$$\begin{aligned} &T^* f(x, x_{n+1}) \\ &\quad = \sup_{\epsilon>0} \left| \int\limits_{|y|>\epsilon} \frac{h(|y|)\Omega(y')}{|y|^n} f\bigl(x - \phi\bigl(|y|\bigr)y', x_{n+1} - \gamma\bigl(|y|\bigr)\bigr)\, dy \right| \\ &\qquad (x, y \in \mathbb{R}^n,\ x_{n+1} \in \mathbb{R},\ y' = y/|y|) \end{aligned}$$

are bounded on $L^p(\mathbb{R}^{n+1})$ for $1 < p < \infty$, $n \geqslant 2$.

Corollary 1. *Let $\gamma \in C^1([0,\infty))$. Suppose that*

(a) *γ is strictly monotone on $[0,\infty)$,*
(b) *γ' is increasing on $(0,\infty)$.*

Then the operators Tf and T^*f (as defined in Theorem 1) are bounded on $L^p(\mathbb{R}^{n+1})$ for $1 < p < \infty$, $n \geqslant 2$.

Corollary 2. *Let $\gamma \in C^1([0,\infty))$. Suppose that*

(a) *$\gamma(0) = 0$ and γ is strictly increasing on $[0,\infty)$,*
(b) *γ' is decreasing on $(0,\infty)$,*
(c) *$t^{-\alpha}\gamma(t)$ is increasing on $(0,\infty)$ for some $\alpha > 0$.*

Then Tf and T^*f (defined in Theorem 1) are bounded on $L^p(\mathbb{R}^{n+1})$ for $1 < p < \infty$, $n \geqslant 2$.

Corollary 3. *Let γ be a nonnegative function on $(0,\infty)$ such that $|\gamma^{(l)}(t)| \geqslant \alpha\gamma(t)/t^l$ for some fixed $l \geqslant 1$ and $\alpha > 0$. (If $l = 1$, then we assume further that $\gamma'(t)$ is monotone.)*

Suppose that either

(a) *γ is strictly increasing and $\gamma(2t) \geqslant \lambda\gamma(t)$ for some fixed $\lambda > 1$, or*

(b) *γ is strictly decreasing, $\gamma(t) \geqslant \lambda\gamma(2t)$ for some fixed $\lambda > 1$, and $\gamma(t) \leqslant c\gamma(2t)$ for some constant $c \geqslant \lambda > 1$.*

Then the operators Tf and T^*f (defined in Theorem 1) are bounded in $L^p(\mathbb{R}^{n+1})$ for $1 < p < \infty$, $n \geqslant 2$.

Theorem 2 [9]. *Let ϕ and Ω be given as in Theorem 1. Let h be a real-valued, measurable, and radial function on $\mathbb{R}^n$, which satisfies the property*

$$\int_0^R \left|h(t)\right|^2 dt \leqslant CR \quad \text{for all } R > 0.$$

Then the singular integral operator

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} f\big(x - \phi\big(|y|\big)y'\big)h\big(|y|\big)\frac{\Omega(y')}{|y|^n}\,dy$$

and its associated maximal operator

$$T^*f(x) = \sup_{\epsilon>0}\left|\int_{|y|>\epsilon} f\big(x - \phi\big(|y|\big)y'\big)h\big(|y|\big)\frac{\Omega(y')}{|y|^n}\,dy\right|$$

are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, $n \geqslant 2$.

Remark. The L^p boundedness of Tf in Theorem 2 was originally obtained by Fan and Pan (see [9]).

Theorem 3. *Let ϕ be a nonnegative C^1 function on $(0, \infty)$. Suppose that either*

(a) *ϕ is strictly increasing and $\phi(2t) \geqslant \lambda\phi(t)$ for some fixed $\lambda > 1$,*
(b) *ϕ' is monotone, and $t^{-\alpha}\phi(t)$ is increasing on $(0, \infty)$ for some fixed $\alpha > 0$,*

or

(c) *ϕ is strictly decreasing, $\phi(t) \geqslant \lambda\phi(2t)$ $(\lambda > 1)$, and $\phi(t) \leqslant c_1\phi(2t)$ $(c_1 \geqslant \lambda)$,*
(d) *ϕ' is monotone, and $|\phi'(t)| \geqslant \alpha\phi(t)/t$ for $t > 0$ and some $\alpha > 0$.*

Then the singular integral operators and their associated maximal operators (as defined in Theorems 1 and 2) are bounded on L^p, $1 < p < \infty$.

Remark. Under closely similar conditions on ϕ (in Theorem 3), the singular integral operator and its associated maximal operator (as defined in Theorem 2) have been proved by other authors (see [10]).

Theorem 4. *Let Ω be given as in Theorem* 1. *Let the functions h and γ, defined on $\mathbb{R}^n$ $(n \geqslant 2)$, be real-valued, measurable, radial and differentiable a.e. on* $[0,\infty)$. *Assume that h is continuous, bounded, and either h is monotone or $h' \in L^1(\mathbb{R})$. Suppose that $|\gamma'(r)|$ is increasing on* $\operatorname{supp}\gamma' \cap [0,\infty)$. *Suppose also that either $\gamma(r)$ is monotone on* $[0,\infty)$ *and $\gamma \in L^\infty(\mathbb{R})$ or $\gamma' \in L^1(\mathbb{R})$. Then the singular integral operator*

$$Tf(x, x_{n+1}) = \text{p.v.} \int f\big(x-y, x_{n+1} - \gamma\big(|y|\big)\big)\, e^{i|y|^{-\beta}}\, \frac{\Omega(y')h(|y|)}{|y|^{n+\alpha}}\, dy$$
$$(x, y \in \mathbb{R}^n,\ x_{n+1} \in \mathbb{R},\ 0 < 2\alpha < \beta)$$

is bounded on $L^2(\mathbb{R}^{n+1})$. Moreover, Tf is bounded on $L^p(\mathbb{R}^{n+1})$ for

$$\frac{\beta}{\beta-\alpha} < p < \frac{\beta}{\alpha} \quad \textit{with } 0 < 2\alpha < \beta,$$

provided that the one-dimensional maximal function $M^\gamma g(x_{n+1})$ (as defined in Theorem 1) *is bounded on $L^p(\mathbb{R})$ for all $p > 1$.*

Corollary 4. *Let $\gamma : [0,\infty) \to \mathbb{R}$ be a measurable C^1 function, which has compact support and is strictly increasing on its compact support. If γ' is increasing on its support, then Tf (in Theorem* 4) *is bounded on $L^p(\mathbb{R}^{n+1})$ for*

$$\frac{\beta}{\beta-\alpha} < p < \frac{\beta}{\alpha} \quad \textit{with } 0 < 2\alpha < \beta.$$

Examples. If γ is convex increasing (such as t^q, $q \geqslant 1$), and has compact domain, then by Corollary 4, the singular integral operator Tf (as defined in Theorem 4) is bounded on L^p for $\beta/(\beta-\alpha) < p < \beta/\alpha$ $(\beta > 2\alpha > 0)$.

If γ is of the type t^q $(q \geqslant 1$ or $q < 0)$, then by Corollary 1, Tf and T^*f (defined in Theorem 1) are bounded on L^p, $1 < p < \infty$. By Corollary 2, similar results hold when γ is of the type t^q, $0 < q < 1$. Also, by Corollary 3, we see that if $\gamma(t) = t^\alpha e^{\beta t}$ $(\alpha > 1, \beta \geqslant 0)$ or $t^{-\alpha} e^{-\beta t}$ $(\alpha > 0,\ \beta \geqslant 0)$, then Tf and T^*f (in Theorem 1) are bounded on L^p, $1 < p < \infty$.

If $\phi(t) = t^q$ $(q \neq 0)$, or $t^\alpha e^{\beta t}$ $(\alpha > 1,\ \beta \geqslant 0)$, or $t^{-\alpha} e^{-\beta t}$ $(\alpha > 0,\ \beta \geqslant 0)$, then by Theorem 3, Tf and T^*f (defined in Theorems 1 or 2) are bounded on L^p, $1 < p < \infty$.

2. Definitions, notations, and preliminaries

We briefly review the space $H^1(S^{n-1})$ (for more details, see [5,6,8]). An "exceptional" atom is an L^∞ function $E(x)$ such that $\|E\|_\infty \leqslant 1$. A "regular" q-atom is an L^q $(1 < q \leqslant \infty)$ function a, which satisfies

(I) $\operatorname{supp}(a) \subset S^{n-1} \cap \{y \in \mathbb{R}^n\colon |y-\zeta| < \rho$ for some $\zeta \in S^{n-1}$ and $\rho \in (0,1]\}$,

(II) $\int_{S^{n-1}} a(\zeta')\,d\sigma(\zeta') = 0$,
(III) $\|a\|_q \leqslant \rho^{(n-1)(1/q-1)}$.

A "q-block" is an $L^q (1 < q \leqslant \infty)$ function, which satisfies the above conditions (I) and (III). Any $\Omega \in H^1(S^{n-1})$ has an atomic decomposition $\Omega = \sum \lambda_j a_j$, where the a_j's are either exceptional atoms or regular q-atoms, and $\sum |\lambda_j| \leqslant C\|\Omega\|_{H^1(S^{n-1})}$ (see [5,6], or [9]).

In particular, if $\Omega \in H^1(S^{n-1})$ satisfies the mean value zero property, then all the atoms a_j in the atomic decomposition of Ω can be chosen to be regular q-atoms for a fixed q, $1 < q \leqslant \infty$.

For the remainder of this paper, we will denote the letter C as a constant, which is not necessarily the same at each occurrence. However, C does not depend on any essential variable. For any $\zeta \in \mathbb{R}^n$, with $\zeta \neq 0$, we write $\zeta/|\zeta| = (\zeta'_1, \zeta'_2, \ldots, \zeta'_n) \equiv (\zeta'_1, \zeta'_*)$. For a fixed $\rho > 0$, we let $A_\rho \zeta = (\rho^2\zeta_1, \rho\zeta_2, \ldots, \rho\zeta_n) \equiv (\rho^2\zeta_1, \rho\zeta_*)$, and let $r \equiv r(\zeta') = |\zeta|^{-1}|A_\rho\zeta|$.

Let $\{a_k\}$ stand for a lacunary sequence of positive real numbers: $a_k > 0$ and $\inf_{k\in\mathbb{Z}}\{a_{k+1}/a_k\} = a > 1$.

We denote $\{\sigma_k\}_{k\in\mathbb{Z}}$ to be a sequence of Borel measures such that $\|\sigma_k\| \leqslant 1$ and $\int d\sigma_k = 0$ for all k. The total variation of σ_k will be denoted by $|\sigma_k|$; and $\sigma^*(f)$ stands for $\sup_{k\in\mathbb{Z}} \||\sigma_k| * f|$.

Also, the sequence $\{\mu_k\}$ denotes sequence of positive Borel measures such that $\|\mu_k\| = 1$ for all k. Given a finite measure μ on $\mathbb{R}^n \equiv \mathbb{R}^m \times \mathbb{R}^{n-m}$ $(1 \leqslant m < n)$, we define another measure $\mu^{(0)}$ in $\mathbb{R}^m$ as $\mu^{(0)}(E) = \mu(E \times \mathbb{R}^{n-m})$ for every Borel subset $E \subset \mathbb{R}^m$; in terms of Fourier transforms, this means $(\mu^{(0)})^\wedge(\zeta^0) = \hat{\mu}(\zeta^0, 0)$. The proofs of Theorems 1–4 rely on the following theorems and lemmas:

Lemma 2.1 [9]. *Let a be a regular ∞-atom on S^{n-1} $(n \geqslant 3)$ with* $\sup(a) \subset S^{n-1} \cap B(\zeta', \rho)$ $(0 < \rho \leqslant 1)$. *Let*

$$F_a(s, \zeta') = (1 - s^2)^{(n-3)/2} \chi_{(-1,1)}(s) \int\limits_{S^{n-2}} a\bigl(s, (1 - s^2)^{1/2}\tilde{y}\bigr)\, d\sigma(\tilde{y}),$$

$$G_a(s, \zeta') = (1 - s^2)^{(n-3)/2} \chi_{(-1,1)}(s) \int\limits_{S^{n-2}} \bigl|a\bigl(s, (1 - s^2)^{1/2}\tilde{y}\bigr)\bigr|\, d\sigma(\tilde{y}).$$

Then up to a constant multiplier independent of a, $F_a(s, \zeta')$ is an ∞-atom on $\mathbb{R}$ and $G_a(s, \zeta')$ is an ∞-block on $\mathbb{R}$. More precisely, there is a constant C independent of a such that

(1) $\operatorname{supp}(F_a) \subseteq (\zeta'_1 - 3r, \zeta'_1 + 3r)$,
(2) $\operatorname{supp}(G_a) \subseteq (\zeta'_1 - 3r, \zeta'_1 + 3r)$,

(3) $\|F_a\|_\infty \leqslant Cr^{-1}$, $\|G_a\|_\infty \leqslant Cr^{-1}$, *and*

$$\int_{\mathbb{R}} F_a(s)\,ds = 0, \quad \text{where } r \equiv r(\zeta') = \left|(\rho^2\zeta_1', \rho\zeta_*')\right|.$$

Lemma 2.2 [9]. *Suppose a is an ∞-atom satisfying* (I)–(III). *The center of the support of a is $\zeta' = (\zeta_1', \zeta_2') \in S^1$. Let*

$$f_a(s,\zeta') = (1-s^2)^{-1/2}\chi_{(-1,1)}(s)\left\{a\left(s, \sqrt{1-s^2}\right) + a\left(s, -\sqrt{1-s^2}\right)\right\},$$
$$g_a(s,\zeta') = (1-s^2)^{-1/2}\chi_{(-1,1)}(s)\left\{\left|a\left(s, \sqrt{1-s^2}\right)\right| + \left|a\left(s, -\sqrt{1-s^2}\right)\right|\right\}.$$

Then up to a constant multiplier independent of a, $f_a(s,\zeta')$ (*respectively, $g_a(s,\zeta')$*) *is a q-atom* (*respectively, q-block*) *on $\mathbb{R}$, where q is any fixed number in the interval* $(1,2)$. *The radius of their support is $r \equiv r(\zeta') = \rho\sqrt{(\rho\zeta_1')^2 + (\zeta_2')^2}$, and the center of their support is ζ_1'.*

Theorem A* [7]. *Suppose that $\mu_k \geqslant 0$ and for some fixed $\alpha > 0$,*

$$\left|\hat{\mu}_k(\zeta) - 1\right| \leqslant C|a_{k+1}A_\rho\zeta|^\alpha, \tag{1}$$

$$\left|\hat{\mu}_k(\zeta)\right| \leqslant C|a_k A_\rho\zeta|^{-\alpha}, \tag{1'}$$

for all $k \in \mathbb{Z}$. Then the maximal operators $Mf(x) = \sup_{k\in\mathbb{Z}} |\mu_k * f(x)|$ is bounded in $L^p(\mathbb{R}^n)$, $1 < p \leqslant \infty$.

Theorem B* [7]. *Suppose that $\|\sigma_k\| \leqslant 1$ and for some fixed $\alpha > 0$,*

$$\left|\hat{\sigma}_k(\zeta)\right| \leqslant C|a_{k+1}A_\rho\zeta|^\alpha, \tag{2}$$

$$\left|\hat{\sigma}_k(\zeta)\right| \leqslant C|a_k A_\rho\zeta|^{-\alpha}, \tag{2'}$$

for all $k \in \mathbb{Z}$, and suppose also that for some $q > 1$,

$$\left\|\sigma^*(f)\right\|_q \leqslant C\|f\|_q, \tag{3}$$

where σ^* is the maximal operator: $\sigma^*(f) = \sup_k \||\sigma_k| * f|$. Then, both

$$Tf(x) = \sum_{k=-\infty}^{\infty} \sigma_k * f(x) \quad \text{and} \quad g(f)(x) = \left(\sum_{k=-\infty}^{\infty} \left|\sigma_k * f(x)\right|^2\right)^{1/2}$$

are bounded operators in $L^p(\mathbb{R}^n)$ for $|1/p - 1/2| < 1/(2q)$.

Theorem C* [7]. *Let $\{\mu_k\}_{k\in\mathbb{Z}}$ be probability measures in $\mathbb{R}^n$ such that*

$$\left|\hat{\mu}_k(\zeta^0, \bar{\zeta}) - \hat{\mu}_k(\zeta^0, 0)\right| \leqslant C|a_{k+1}A_\rho\bar{\zeta}|^\alpha, \tag{4}$$

$$\left|\hat{\mu}_k(\zeta^0, \bar{\zeta})\right| \leqslant C|a_k A_\rho\bar{\zeta}|^{-\alpha}. \tag{4'}$$

Suppose that $M^0 g(x^0) = \sup_k |\mu_k^{(0)} * g(x^0)|$ *is a bounded operator in* $L^p(\mathbb{R}^m)$ $(1 \leqslant m < n)$ *for all* $p > 1$. *Then,* $Mf(x) = \sup_{k\in\mathbb{Z}} |\mu_k * f(x)|$ *is also bounded in* $L^p(\mathbb{R}^n)$ *for all* $p > 1$.

Theorem D* [7]. *Suppose that* $\|\sigma_k\| \leqslant 1$ *and that the measures* $\{\sigma_k\}_{k\in\mathbb{Z}}$ *satisfy the same estimates* (4), (4′) *required for* $\{\mu_k\}_{k\in\mathbb{Z}}$ *in Theorem* C*, *and also either*

$$\left|\hat{\sigma}_k(\zeta^0, 0)\right| \leqslant C|b_{k+1}\zeta^0|^{\alpha}, \tag{5a}$$

$$\left|\hat{\sigma}_k(\zeta^0, 0)\right| \leqslant C|b_k\zeta^0|^{-\alpha}, \tag{5′a}$$

or

$$\left|\hat{\sigma}_k(\zeta^0, 0)\right| \leqslant C|b_{k+1}A_\rho\zeta^0|^{\alpha}, \tag{5b}$$

$$\left|\hat{\sigma}_k(\zeta^0, 0)\right| \leqslant C|b_kA_\rho\zeta^0|^{-\alpha}, \tag{5′b}$$

where $\{b_k\}_{k\in\mathbb{Z}}$ *is another lacunary sequence of positive numbers. If* $\sigma^*(f) = \sup_{k\in\mathbb{Z}} ||\sigma_k| * f|$ *and* $\sigma^*_{(0)}(g) = \sup_{k\in\mathbb{Z}} \|\sigma_k^{(0)}| * g|$ *are bounded in* $L^q(\mathbb{R}^n)$ *and* $L^q(\mathbb{R}^m)$, *respectively, then* Tf *and* $g(f)$ (*as defined in Theorem* B*) *are bounded in* $L^p(\mathbb{R}^n)$ *for* $|1/p - 1/2| < 1/(2q)$.

Theorem D′* [7]. *Suppose that* $\|\sigma_k\| \leqslant 1$ *and that the measures* $\{\sigma_k\}_{k\in\mathbb{Z}}$ *satisfy the estimates*

$$\begin{aligned} &\hat{\sigma}_k(\zeta^0, 0) = 0 \quad \text{for all } k \in \mathbb{Z}, \\ &\hat{\sigma}_k(\zeta^0, \bar{\zeta}) \leqslant C\min\left\{|a_{k+1}A_\rho\bar{\zeta}|^{\alpha},\ |a_kA_\rho\bar{\zeta}|^{-\alpha}\right\}. \end{aligned} \tag{6}$$

If $\sigma^*(f) = \sup_{k\in\mathbb{Z}} \|\sigma_k| * f|$ *and* $\sigma_0^*(g) = \sup_{k\in\mathbb{Z}} \|\sigma_k^{(0)}| * g|$ *are bounded in* $L^q(\mathbb{R}^n)$ *and* $L^q(\mathbb{R}^m)$, *respectively, then* Tf *and* $g(f)$ (*defined in Theorem* B*) *are bounded in* $L^p(\mathbb{R}^n)$ *for* $|1/p - 1/2| < 1/(2q)$.

Theorem E* [7]. *Let* σ_k *be Borel measures supported in* $\{x \in \mathbb{R}^n\colon\ |x| < a_{k+1}\}$ (*respectively,* $\{x \in \mathbb{R}^n\colon\ |\bar{x}| < a_{k+1}\}$) *verifying the hypotheses of Theorem* B* (*respectively, Theorem* D*) *for all* $q > 1$. *Then* T^* *is bounded in* L^p, $1 < p < \infty$. *Here* $T^*f(x) = \sup_{k\in\mathbb{Z}} |T_kf(x)|$, *with* $T_kf(x) = \sum_{j=k}^{\infty} \sigma_j * f(x)$.

Remarks. 1. Theorems A*–E* are the modified versions of Theorems A–E in [7].

2. In Theorems A*–E*, ρ is any fixed positive number. All the bounds in Theorems A*–E* are independent of ρ.

We now proceed to prove Theorems A*–E* before we prove Theorems 1–4.

3. Proofs of Theorems A*–E*

The proofs of these theorems are essentially the same as those in [7], except some minor changes. We refer the reader to read the proofs of Theorems A–E in [7], with the following modifications.

The Schwartz functions Φ_k appearing in Theorems A–D′ [7] should be replaced as follows: Let ϕ be the Gauss–Weierstrass kernel on $\mathbb{R}$, i.e., $\phi(t) = e^{-\pi t^2}$, $t \in \mathbb{R}$. Note that the Fourier transform of ϕ is the function itself.

Define Φ_k on $\mathbb{R}^n$ (respectively, $\mathbb{R}^{n-m}$) by $\widehat{\Phi}_k(\zeta) = \hat{\phi}(a_k|A_\rho\zeta|)$ (respectively, $\widehat{\Phi}_k(\bar{\zeta}) = \hat{\phi}(a_k|A_\rho\bar{\zeta}|)$).

The terms $a_k|\zeta|$ (respectively $a_k|\bar{\zeta}|$) appearing in Theorems A–D′ [7] should be replaced by $a_k|A_\rho\zeta|$ (respectively, $a_k|A_\rho\bar{\zeta}|$). We replace Δ_j in Theorem B [7] by

$$\Delta_j = \left\{\zeta\colon a_{j+1}^{-1} \leqslant |A_\rho\zeta| \leqslant a_{j-1}^{-1}\right\}.$$

In Theorem A [7], replace f^* by $M_1^H \circ M_2^H \circ \cdots \circ M_n^H(f)$, where $M_i^H(f)(x)$ is the one-dimensional Hardy–Littlewood maximal function acting on the ith coordinate of the x-variable. The phrase "…the Hardy–Littlewood operator acting on the $\bar{x}$-variable" in the proofs of Theorems C–D′ (see lines 12–13, p. 547 of [7]) should be replaced by "…the composition of one-dimensional Hardy–Littlewood maximal operators acting on the coordinates of the $\bar{x}$-variable."

In the modification of Theorem D, we remark that if $|\hat{\sigma}_k(\zeta^0, 0)|$ satisfies (5a) and (5′a), then by an application of Theorem D′ [7], $T^{(1)}f$ and $g^{(1)}(f)$ are bounded in $L^p(\mathbb{R}^n)$, $|1/p - 1/2| < 1/2q$ (see line 21, p. 547 of [7]). Otherwise, if $|\hat{\sigma}_k(\zeta^0, 0)|$ satisfies (5b) and (5′b) instead of (5a) and (5′a), then the same results still hold by an application of Theorem D′*. We now lay out the proof of Theorem E* in detail.

Proof of Theorem E*. We take a radial Schwartz function ϕ such that $\phi(\zeta) = 1$ when $|\zeta| < a^{-1}$ and $\phi(\zeta) = 0$ when $|\zeta| > a$. Recall that the number a comes from the lacunary sequence $\{a_k\}$. Define Φ_k by $\widehat{\Phi}_k(\zeta) = \phi(a_k|A_p\zeta|)$. Write $T_k f$ as

$$T_k f = \Phi_k * \left(Tf - \sum_{j=-\infty}^{k-1} \sigma_j * f\right) + (\delta - \Phi_k) * \sum_{j=k}^{\infty} \sigma_j * f.$$

Observe that $|\Phi_k * Tf(x)| \leqslant CM^H Tf(x)$ for all $k \in \mathbb{Z}$. By Theorem B*, $\sup_{k\in\mathbb{Z}} |\Phi_k * Tf(x)|$ is bounded on L^p, $1 < p < \infty$. Now, write

$$\sup_{k\in\mathbb{Z}} \left|\Phi_k * \sum_{j=-\infty}^{k-1} \sigma_j * f\right| \leqslant \sum_{j=1}^{\infty} \sup_{k\in\mathbb{Z}} |\sigma_{k-j} * \Phi_k * f|,$$

where each summand in the sum above is bounded on L^p, because of the boundedness of σ^*. Moreover, each term in the sum above has an L^2-norm of the order $a^{-\alpha j}$. To see this, we write

$$\sup_{k\in\mathbb{Z}} |\sigma_{k-j} * \Phi_k * f| \leqslant \left(\sum_{k=-\infty}^{\infty} |\sigma_{k-j} * \Phi_k * f|^2\right)^{1/2};$$

by Pancherel theorem, it suffices to show that

$$\sum_{k=-\infty}^{\infty} \left|\widehat{\Phi}_k(\zeta)\hat{\sigma}_{k-j}(\zeta)\right|^2 \leqslant Ca^{-2\alpha j}.$$

There exists an $l \in \mathbb{Z}$ such that $a_{l+1}^{-1} \leqslant |A_\rho \zeta| \leqslant a_l^{-1}$ for $\zeta \neq 0$. Using the support conditions on ϕ and hypothesis for σ, we have

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \left|\widehat{\Phi}_k(\zeta)\hat{\sigma}_{k-j}(\zeta)\right|^2 &\leqslant C \sum_{k=-\infty}^{l+1} \left(a_{k-j+1}a_l^{-1}\right)^{2\alpha} \\ &\leqslant C \sum_{k=-\infty}^{l+1} a^{-2\alpha(j-1)}a^{2\alpha(k-l)} \leqslant Ca^{-2\alpha j}.\end{aligned}$$

By interpolating the L^2-norm of the order $a^{-\alpha j}$ with an L^{p_0}-norm ($p_0 > p$), we get a factor $a^{-\epsilon j}$ in the L^p-norm, which makes the sum $\sum_{j=1}^{\infty} \sup_{k\in\mathbb{Z}} |\sigma_{k-j} * \Phi_k * f|$ converges. Using similar arguments, we see that $\sup_{k\in\mathbb{Z}} |(\delta - \Phi_k) * \sum_{j=k}^{\infty} \sigma_j * f|$ is bounded in L^p, $1 < p < \infty$. Theorem E* is proved.

4. Proof of Theorem 1

In view of the atomic decomposition of Ω, it suffices to show that the singular integral operator

$$T_a f(x, x_{n+1}) = \int \frac{h(|y|)a(y')}{|y|^n} f\big(x - \phi(|y|)y', x_{n+1} - \gamma(|y|)\big)\, dy$$

and its associated maximal operator

$$T_a^* f(x, x_{n+1}) = \sup_{\epsilon>0} \left| \int_{|y|>\epsilon} \frac{h(|y|)a(y')}{|y|^n} f\big(x - \phi(|y|)y', x_{n+1} - \gamma(|y|)\big)\, dy \right|$$

are bounded on $L^p(\mathbb{R}^{n+1})$, $1 < p < \infty$, $n \geqslant 2$, with the bounds independent of the regular ∞-atom $a(y')$. We may assume w.o.l.g. that $\operatorname{supp}(a) \subset B(\mathbf{1}, \rho) \cap S^{n-1}$, where $\mathbf{1} = (1, 0, \ldots, 0)$. The proof requires applications of Theorems C*, D$'^*$, and E*, and Lemmas 2.1 and 2.2 in [9]. We will prove for the case $n \geqslant 3$, since the case $n=2$ is similar (using Lemma 2.2 [9] instead of Lemma 2.1 [9]). We write $T_a f = \sum_k \sigma_k * f$, where

$$\hat{\sigma}_k(\zeta,\zeta_{n+1}) = \int_{|y|\simeq 2^k} e^{i\phi(|y|)|\zeta|\zeta'\cdot y'} e^{i\zeta_{n+1}\gamma(|y|)} \frac{h(|y|)a(y')}{|y|^n}\,dy$$
$$(\zeta, y \in \mathbb{R}^n,\ \zeta_{n+1}\in\mathbb{R},\ \zeta' = \zeta/|\zeta|,\ y' = y/|y|).$$

Then $\{\sigma_k\}$ are finite Borel measures with $\|\sigma_k\| \leqslant C$, $\int d\sigma_k = 0$, and $\hat{\sigma}_k(0,\zeta_{n+1}) = 0$ for all $k\in\mathbb{Z}$.

Let $\mu_k = |\sigma_k|$ be the total variation of σ_k. In terms of Fourier transform,

$$\hat{\mu}_k(\zeta,\zeta_{n+1}) = \int_{|y|\simeq 2^k} e^{i\phi(|y|)|\zeta|\zeta'\cdot y'} e^{i\zeta_{n+1}\gamma(|y|)} \left|\frac{h(|y|)a(y')}{|y|^n}\right| dy.$$

Define the measures $\mu_k^{(0)}$ by $\hat{\mu}_k^{(0)}(\zeta_{n+1}) = \hat{\mu}_k(0,\zeta_{n+1})$. Then

$$\mu_k^{(0)} * g(x_{n+1}) = \int_{|y|\simeq 2^k} g\bigl(x_{n+1} - \gamma\bigl(|y|\bigr)\bigr)\left|\frac{h(|y|)a(y')}{|y|^n}\right| dy.$$

We need to show that $\hat{\sigma}_k$ satisfy (6), $\hat{\mu}_k$ satisfy (4) and (4′), and that $\sup_k |\mu_k^{(0)} * g(x_{n+1})|$ is bounded on $L^p(\mathbb{R})$ for all $p>1$.

For $\zeta \neq 0$, we choose a rotation θ such that $\theta(\zeta) = |\zeta|\mathbf{1} = |\zeta|(1,0,0,\dots,0)$. Let $y' = (s, y_2', \dots, y_n')$. Then

$$\hat{\sigma}_k(\zeta,\zeta_{n+1}) = \int_{2^k}^{2^{k+1}} e^{i\zeta_{n+1}\gamma(t)}\frac{h(t)}{t} \times \int_{S^{n-1}} a\bigl(\theta^{-1}(y')\bigr) e^{i|\zeta|\phi(t)\langle\zeta',\theta^{-1}(y')\rangle}\,d\sigma(y')\,dt,$$

where $a(\theta^{-1}(y'))$ is again a regular ∞-atom with support in $B(\zeta',\rho)\cap S^{n-1}$, $\zeta' = \zeta/|\zeta|$. Thus

$$\hat{\sigma}_k(\zeta,\zeta_{n+1}) = \int_{2^k}^{2^{k+1}} \frac{h(t)}{t} e^{i\zeta_{n+1}\gamma(t)}\biggl(\int e^{i|\zeta|\phi(t)s} F_a(s,\zeta')\,ds\biggr)dt,$$

where $F_a(s,\zeta')$ has support in $(\zeta_1' - 3r, \zeta_1' + 3r)$ $(r \equiv r(\zeta'))$ (see Lemma 2.1 [9]). Also,

$$\hat{\mu}_k(\zeta,\zeta_{n+1}) = \int_{2^k}^{2^{k+1}} \frac{|h(t)|}{t} e^{i\zeta_{n+1}\gamma(t)}\biggl(\int e^{i|\zeta|\phi(t)s} G_a(s,\zeta')\,ds\biggr)dt.$$

By the cancellation property of $F_a(s,\zeta')$,

$$\begin{aligned}
\left|\hat\sigma_k(\zeta,\zeta_{n+1})\right| &= \left|\int_{2^k}^{2^{k+1}} \frac{h(t)}{t} e^{i\zeta_{n+1}\gamma(t)} \int \left(e^{i|\zeta|\phi(t)s}-1\right) F_a(s,\zeta')\,ds\,dt\right| \\
&\leqslant \left\{\int_{2^k}^{2^{k+1}} |\zeta|\left|h(t)\phi(t)\right| \frac{dt}{t}\right\}\left\{\int \left|sF_a(s,\zeta')\right| ds\right\} \\
&\leqslant C|2^{dk}\zeta|r^{-1} \int_{\zeta_1'-3r}^{\zeta_1'+3r} |s|\,ds \\
&\leqslant C|2^{dk}r\zeta| = C|2^{dk}A_\rho\zeta|. \qquad (7)
\end{aligned}$$

The last inequality follows due to a simple change of variable $s \to s+\zeta_1'$ in the integration. On the other hand, by Hölder's inequality,

$$\begin{aligned}
\left|\hat\sigma_k(\zeta,\zeta_{n+1})\right|^2 &\leqslant \left\{\int_{2^k}^{2^{k+1}} \left|h(t)e^{i\zeta_{n+1}\gamma(t)}\right|^2 \frac{dt}{t}\right\} \\
&\quad\times \left\{\int_{2^k}^{2^{k+1}} \left|\int e^{i|\zeta|\phi(t)s} F_a(s,\zeta')\,ds\right|^2 \frac{dt}{t}\right\} \\
&\leqslant C \iint \left\{\int_{2^k}^{2^{k+1}} e^{i|\zeta|\phi(t)(s-\tilde s)} \frac{dt}{t}\right\} F_a(s,\zeta')\overline{F}_a(\tilde s,\zeta')\,ds\,d\tilde s. \qquad (8)
\end{aligned}$$

Denote

$$I_k(|\zeta|) = \int_{2^k}^{2^{k+1}} e^{i|\zeta|\phi(t)(s-\tilde s)} \frac{dt}{t}. \qquad (9)$$

It is clear that $|I_k(|\zeta|)| \leqslant \ln 2$. Also by integrating by parts,

$$I_k\big(|\zeta|\big) = \frac{e^{i|\zeta|\phi(t)(s-\tilde s)}}{i|\zeta|(s-\tilde s)\phi'(t)t}\Bigg|_{2^k}^{2^{k+1}} + \int_{2^k}^{2^{k+1}} \frac{e^{i|\zeta|\phi(t)(s-\tilde s)}}{i|\zeta|(s-\tilde s)} \left\{\frac{\phi'(t)+t\phi''(t)}{[t\phi'(t)]^2}\right\} dt.$$

From hypotheses (b) and (c) in Theorem 1, we find that $|I_k(|\zeta|)| \leqslant C|\zeta(s-\tilde s)2^{dk}|^{-1}$. Therefore, $|I_k(|\zeta|)| \leqslant C\min\{1, |\zeta(s-\tilde s)2^{dk}|^{-1}\}$ which implies that $|I_k(|\zeta|)| \leqslant C|\zeta(s-\tilde s)2^{dk}|^{-1/2}$. So

$$\begin{aligned}\left|\hat{\sigma}_k(\zeta,\zeta_{n+1})\right|^2 &\leqslant C|2^{dk}\zeta|^{-1/2}\iint |s-\tilde{s}|^{-1/2}\left|F_a(s,\zeta')\overline{F}_a(\tilde{s},\zeta')\right|ds\,d\tilde{s}\\ &\leqslant C|2^{dk}\zeta|^{-1/2}r^{-1}\int\left(\int_{\zeta_1'-3r}^{\zeta_1'+3r}|s-\tilde{s}|^{-1/2}\,ds\right)\left|\overline{F}_a(\tilde{s},\zeta')\right|d\tilde{s}.\end{aligned}$$

After a change of variable, the inner integral above becomes $\int_{\zeta_1'-3r-\tilde{s}}^{\zeta_1'+3r-\tilde{s}}|s|^{-1/2}\,ds$, which is dominated by $\int_{-6r}^{6r}|s|^{-1/2}\,ds\leqslant Cr^{1/2}$.

Thus

$$\left|\hat{\sigma}_k(\zeta,\zeta_{n+1})\right|\leqslant C|2^{dk}r\zeta|^{-1/4}\|\overline{F_a}\|_1^{1/2}\leqslant C|2^{dk}r\zeta|^{-1/4}=C|2^{dk}A_\rho\zeta|^{-1/4}.$$

Consequently,

$$\left|\hat{\sigma}_k(\zeta,\zeta_{n+1})\right|\leqslant C\min\left\{|2^{dk}A_\rho\zeta|,|2^{dk}A_\rho\zeta|^{-1/4}\right\}.$$

By the same token, we obtain

$$\left|\hat{\mu}_k(\zeta,\zeta_{n+1})-\hat{\mu}_k(0,\zeta_{n+1})\right|\leqslant C|2^{kd}A_\rho\zeta|, \tag{4}$$

and

$$\left|\hat{\mu}_k(\zeta,\zeta_{n+1})\right|\leqslant C|2^{dk}A_\rho\zeta|^{-1/4}. \tag{4$'$}$$

It remains to show that $\sup_k|\mu_k^{(0)}*g(x_{n+1})|$ is bounded on $L^p(\mathbb{R})$ for all $p>1$. But

$$\begin{aligned}\left|\mu_k^{(0)}*g(x_{n+1})\right| &= \left|\|a\|_1\int_{2^k}^{2^{k+1}}g\bigl(x_{n+1}-\gamma(t)\bigr)\frac{|h(t)|}{t}\,dt\right|\\ &\leqslant C\frac{1}{2^k}\int_{2^k}^{2^{k+1}}\left|g\bigl(x_{n+1}-\gamma(t)\bigr)\right|dt\\ &\leqslant CM^\gamma g(x_{n+1})\quad\text{for all } k\in\mathbb{Z}.\end{aligned}$$

Thus $\sup_k|\mu_k^{(0)}*g(x_{n+1})|\leqslant CM^\gamma g(x_{n+1})$, which is bounded on $L^p(\mathbb{R})$ for all $p>1$ (by hypothesis). Therefore, by Theorems D′* and C*, we have $\|T_af\|_p\leqslant C\|f\|_p$ for $1<p<\infty$. Observe that

$$\begin{aligned}T_a^*f(x) &= \sup_{\epsilon>0}\left|\int_{|y|>\epsilon}\frac{h(|y|)a(y')}{|y|^n}f\bigl(x-\phi\bigl(|y|\bigr)y',x_{n+1}-\gamma\bigl(|y|\bigr)\bigr)\,dy\right|\\ &\leqslant \sup_k\left|\sum_{j=k}^{\infty}\sigma_j*f(x)\right|+\sup_k\bigl||\sigma_k|*f(x)\bigr|.\end{aligned}$$

We have shown that the second term on the RHS of the above inequality is bounded on L^p for $1 < p < \infty$. Since $\operatorname{supp}(\sigma_k) \subset \{(x, x_{n+1})\colon |x| < 2^{k+1}\}$, we may apply Theorem E* to conclude that the first term above is also bounded on L^p, $1 < p < \infty$. Hence, $\|T_a^* f\|_p \leqslant C\|f\|_p$ for $1 < p < \infty$. Theorem 1 is proved. □

Remark. When $n = 2$, F_a (respectively, G_a) in the proof of Theorem 1 is replaced by f_a (respectively g_a). By Lemma 2.2 [9], f_a (respectively, g_a) is a q-atom (respectively, q-block), say $q = 3/2$. Then the second integral in (7) becomes

$$\begin{aligned}\int_{\zeta_1'-r}^{\zeta_1'+r} \bigl|sf_a(s,\zeta')\bigr|\,ds &\leqslant \|f_a\|_{3/2}\left\{\int_{\zeta_1'-r}^{\zeta_1'+r} |s|^3\,ds\right\}^{1/3} \\ &\leqslant Cr^{-1/3}\cdot r^{4/3} = Cr \quad (r \equiv r(\zeta')),\end{aligned}$$

so that $|\hat{\sigma}_k(\zeta, \zeta_{n+1})| \leqslant C|2^{dk} r\zeta| = C|2^{dk} A_\rho \zeta|$. Also from (9), $|I_k(|\zeta|)| \leqslant C|\zeta(s-\tilde{s})2^{dk}|^{-1/6}$. Thus (8) becomes

$$\begin{aligned}\bigl|\hat{\sigma}_k(\zeta,\zeta_{n+1})\bigr|^2 &\leqslant C|2^{dk}\zeta|^{-1/6} \\ &\quad\times \int\left(\int |s-\tilde{s}|^{-1/6}\bigl|f_a(s,\zeta')\bigr|\,ds\right)\bigl|\bar{f}_a\bigl(\tilde{s},\zeta'\bigr)\bigr|\,d\tilde{s} \\ &\leqslant C|2^{dk}\zeta|^{-1/6}\|f_a\|_q \\ &\quad\times \int\left\{\int |s-\tilde{s}|^{-1/2}\,ds\right\}^{1/3}\bigl|\bar{f}_a(\tilde{s},\zeta')\bigr|\,d\tilde{s} \\ &\leqslant C|2^{dk}\zeta|^{-1/6} r^{-1/3}\cdot r^{1/6}\|\bar{f}_a\|_1 \leqslant C|2^{dk} r\zeta|^{-1/6},\end{aligned}$$

whence $|\hat{\sigma}_k(\zeta,\zeta_{n+1})| \leqslant C|2^{dk}A_\rho\zeta|^{-1/12}$.

Proof of Corollaries 1–3. It is enough to show that the one-dimensional maximal function $M^\gamma g(x_{n+1})$ is bounded on $L^p(\mathbb{R})$ for all $p > 1$. For the proof of Corollaries 1–2, see [15]. To prove Corollary 3, note that

$$M^\gamma g(x_{n+1}) \leqslant 3\sup_{k\in\mathbb{Z}}\left\{\frac{1}{2^k}\int_{2^k}^{2^{k+1}} \bigl|g\bigl(x-\gamma(t)\bigr)\bigr|\,dt\right\}.$$

Thus it suffices to show that the latter maximal function above is bounded on $L^p(\mathbb{R})$ for all $p > 1$. An easy application of Theorem A [7] and van der Corput's lemma will yield the results.

5. Proof of Theorem 2

We will show that the singular integral operator

$$T_a f(x) = \text{p.v.} \int_{\mathbb{R}^n} f\big(x - \phi\big(|y|\big)y'\big)h\big(|y|\big)a(y')|y|^{-n}\,dy \quad (n \geqslant 2)$$

and its associated maximal operator

$$T_a^* f(x) = \sup_{\epsilon>0}\left| \int_{|y|>\epsilon} f\big(x - \phi\big(|y|\big)y'\big)h\big(|y|\big)a(y')|y|^{-n}\,dy\right|$$

are bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, $n \geqslant 2$, with the bounds independent of the regular ∞-atom $a(y')$. Similar to the proof of Theorem 1, we write $T_a f(x) = \sum_k \sigma_k * f(x)$, with

$$\hat{\sigma}_k(\zeta) = \int_{2^k}^{2^{k+1}} \frac{h(t)}{t} \int e^{i|\zeta|\phi(t)s} F_a(s, \zeta')\,ds\,dt.$$

Let $\mu_k = |\sigma_k|$ be the total variation of σ_k. In terms of Fourier transforms,

$$\hat{\mu}_k(\zeta) = \int_{2^k}^{2^{k+1}} \frac{|h(t)|}{t} \int e^{i|\zeta|\phi(t)s} G_a(s, \zeta')\,ds\,dt.$$

To prove the boundedness of $T_a f$, we will apply Theorems A* and B*. That is, we need to show that $\hat{\sigma}_k(\zeta)$ satisfy (2) and (2′), and $\hat{\mu}_k(\zeta)$ satisfy (1) and (1′). By the cancellation property of $F_a(s, \zeta')$,

$$\begin{aligned}\big|\hat{\sigma}_k(\zeta)\big| &\leqslant \left\{\int_{2^k}^{2^{k+1}} |\zeta|\big|\phi(t)h(t)\big|\frac{dt}{t}\right\}\left\{\int_{\zeta_1'-3r}^{\zeta_1'+3r} \big|sF_a(s, \zeta')\big|\,ds\right\} \\ &\leqslant C|2^{dk}r\zeta| = C|2^{dk}A_\rho\zeta|.\end{aligned}$$

Also by Hölder's inequality,

$$\begin{aligned}\big|\hat{\sigma}_k(\zeta)\big| &\leqslant \left\{\int_{2^k}^{2^{k+1}} \big|h(t)\big|^2\frac{dt}{t}\right\}^{1/2}\left\{\int_{2^k}^{2^{k+1}} \left|\int e^{i|\zeta|\phi(t)s}F_a(s, \zeta')\,ds\right|^2\frac{dt}{t}\right\}^{1/2} \\ &\leqslant C|2^{dk}r\zeta|^{-1/4} = C|2^{dk}A_\rho\zeta|^{-1/4}.\end{aligned}$$

By the same token, we have

$$\big|\hat{\mu}_k(\zeta) - \hat{\mu}_k(0)\big| \leqslant C|2^{dk}A_\rho\zeta|$$

and

$$\left|\hat{\mu}_k(\zeta)\right| \leqslant C|2^{dk}A_\rho\zeta|^{-1/4}.$$

Consequently, we may apply Theorems A* and B* (respectively, Theorem E*) to conclude that $T_a f$ (respectively, $T_a^* f$) are bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, $n \geqslant 2$. Theorem 2 is proved. □

6. Proof of Theorem 3

We will prove the L^p boundedness for the singular integral $Tf(x, x_{n+1})$ in Theorem 1. Assume that ϕ satisfies conditions (a) and (b) in Theorem 3. By inspection of the proof in Theorem 1, it suffices to show that for all $k \in \mathbb{Z}$,

$$\left|\hat{\sigma}_k(\zeta, \zeta_{n+1})\right| \leqslant C \min\left\{|a_{k+1}A_\rho\zeta|,\ |a_k A_\rho\zeta|^{-1/4}\right\},$$

where $\{a_k\} \equiv \{\phi(2^k)\}$ is a lacunary sequence. That $|\hat{\sigma}_k(\zeta, \zeta_{n+1})| \leqslant C|\phi(2^{k+1})A_\rho\zeta|$ is clear. For the remaining part, it is enough to show that

$$\left|I_k\big(|\zeta|\big)\right| \equiv \left|\int_{2^k}^{2^{k+1}} e^{i|\zeta|(s-\tilde{s})\phi(t)}\frac{dt}{t}\right| \leqslant C\left|\zeta(s-\tilde{s})\phi(2^k)\right|^{-1}$$

(see Eq. (9)). By a change of variable,

$$I_k\big(|\zeta|\big) = \int_1^2 e^{i|\zeta|(s-\tilde{s})\phi(2^k t)}\frac{dt}{t} \equiv \int_1^2 g'(t)\frac{dt}{t},$$

where

$$g(t) = \int_1^t e^{i|\zeta|(s-\tilde{s})\phi(2^k u)}\,du, \quad 1 \leqslant t \leqslant 2.$$

Using conditions (a) and (b) on ϕ, we obtain

$$\frac{d}{du}\phi(2^k u) = 2^k\phi'(2^k u) \geqslant \alpha\frac{\phi(2^k u)}{u} \geqslant \alpha\frac{\phi(2^k)}{t} \quad \text{for } 1 \leqslant u \leqslant t \leqslant 2.$$

Thus by van der Corput's lemma, $|g(t)| \leqslant \alpha^{-1}|\zeta(s-\tilde{s})\phi(2^k)|^{-1}t$ for $1 \leqslant t \leqslant 2$. Hence by integration by parts, $|I_k(|\zeta|)| \leqslant C|\zeta(s-\tilde{s})\phi(2^k)|^{-1}$.

If instead ϕ satisfies conditions (c) and (d) in Theorem 3, then by similar arguments, we get

$$\left|\hat{\sigma}_{-k}(\zeta, \zeta_{n+1})\right| \leqslant C \min\left\{|b_{k+1}A_\rho\zeta|, |b_k A_\rho\zeta|^{-1/4}\right\}$$

for all $k \in \mathbb{Z}$. Here the lacunary sequence $\{b_k\}$ is defined by $b_k = \phi(2^{-k})$, $k \in \mathbb{Z}$. Theorem 3 is proved □

7. Proof of Theorem 4

Since h is bounded, we may assume w.o.l.g. that $h \geqslant 0$. Again, it suffices to consider the regular ∞-atom a in place of Ω. First we show the L^2-boundedness of Tf via Plancherel's theorem. Next, we define a family of analytic operators $T_z f$ by

$$T_z f(x, x_{n+1}) = \text{p.v.} \int_{\mathbb{R}^n} f\big(x - y, x_{n+1} - \gamma\big(|y|\big)\big) \frac{e^{i|y|^{-\beta}} a(y') h(|y|)}{|y|^{n+\alpha+z}}\, dy.$$

We then show that for $z = \sigma + i\tau$,

$$\|T_z f\|_2 \leqslant C\big(1 + |z|\big)\|f\|_2 \quad \text{for } -\alpha < \sigma < \frac{\beta - 2\alpha}{2},\ \tau \in \mathbb{R},$$

and $\|T_z f\|_p \leqslant C\|f\|_p$, $1 < p < \infty$, for $\sigma = -\alpha$ and $\tau \in \mathbb{R}$. Finally, an application of analytic interpolation theorem will yield the results. The L^2-boundedness of Tf and $T_z f$ have been shown in [16]. It remains to show that for $z = \sigma + i\tau$, with $\sigma = -\alpha$ and $\tau \in \mathbb{R}$,

$$\|T_z f\|_p \leqslant C\|f\|_p, \quad 1 < p < \infty.$$

We will prove this for the case $n \geqslant 3$, since the case $n = 2$ is essentially the same. Write $T_z f(x, x_{n+1}) = \sum_k \sigma_k * f(x, x_{n+1})$, where

$$\begin{aligned}
\hat{\sigma}_k(\zeta, \zeta_{n+1}) &= \int_{|y| \simeq 2^k} e^{i\zeta\cdot y} e^{i\zeta_{n+1}\gamma(|y|)} e^{i|y|^{-\beta}} \frac{h(|y|)\, a(y')}{|y|^{n+\alpha+z}}\, dy \\
&= \int_{|y| \simeq 2^k} e^{i\zeta\cdot y} e^{i\zeta_{n+1}\gamma(|y|)} e^{i|y|^{-\beta}} \frac{h(|y|) a(y')}{|y|^{n+i\tau}}\, dy.
\end{aligned}$$

Let $\mu_k = |\sigma_k|$ be the total variation of σ_k. In terms of Fourier transform,

$$\hat{\mu}_k(\zeta, \zeta_{n+1}) = \int_{|y| \simeq 2^k} e^{i\zeta\cdot y} e^{i\zeta_{n+1}\gamma(|y|)} \frac{|h(|y|) a(y')|}{|y|^n}\, dy.$$

Define the measures $\mu_k^{(0)}$ by $\hat{\mu}_k^{(0)}(\zeta_{n+1}) = \hat{\mu}_k(0, \zeta_{n+1})$. Note that

$$\mu_k^{(0)} * g(x_{n+1}) = \int_{|y| \simeq 2^k} g\big(x_{n+1} - \gamma\big(|y|\big)\big) \frac{|h(|y|) a(y')|}{|y|^n}\, dy.$$

Similar to the proof in Theorem 1, we see that the measures σ_k satisfy the estimate (6), and the measures μ_k satisfy the estimates (4) and (4′). Also, $\sup_k |\mu_k^{(0)} * g(x_{n+1})| \leqslant CM^\gamma g(x_{n+1})$. Thus, by Theorems D′* and C*, we have

$$\|T_z f\|_p \leqslant C\|f\|_p, \quad 1 < p < \infty,\ z = -\alpha + i\tau,\ \tau \in \mathbb{R}.$$

Theorem 4 is proved. □

Proof of Corollary 4. See [15].

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